A simple method for determining the spatial resolution of a general inverse problem

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ABSTRACT

The resolution matrix of an inverse problem defines a linear relationship in which each solution parameter is derived from the weighted averages of nearby true-model parameters, and the resolution matrix elements are the weights. Resolution matrices are not only widely used to measure the solution obtainability or the inversion perfectness from the data based on the degree to which the matrix approximates the identity matrix, but also to extract spatial resolution or resolution length information. Resolution matrices presented in previous spatial-resolution analysis studies can be divided into three classes: direct resolution matrix, regularized/stabilized resolution matrix, and hybrid resolution matrix. The direct resolution matrix can yield resolution length information only for ill-posed inverse problems. The regularized resolution matrix cannot give any spatial-resolution information. The hybrid resolution matrix can provide resolution length information; however, this depends on the regularization contribution to the inversion. The computation of the matrices needs matrix operation, however, and this is often a difficult problem for very large inverse problems. Here, a new class of resolution matrices, generated using a Gaussian approximation (called the
statistical resolution matrices), is proposed whereby the direct determination of the matrix is accomplished via a simple one-parameter nonlinear inversion performed based on limited pairs of random synthetic models and their inverse solutions. Tests showed that a statistical resolution matrix could not only measure the resolution obtainable from the data, but also provided reasonable spatial/temporal resolution or resolution length information. The estimates were restricted to forward/inversion processes and were independent of the degree of inverse skill used in the solution inversion; therefore, the original inversion codes did not need to be modified. The absence of a requirement for matrix operations during the estimation process indicated that this approach is particularly suitable for very large linear/linearized inverse problems. The estimation of statistical resolution matrices is useful for both direction-dependent and direction-independent resolution estimations. Interestingly, even a random synthetic input model without specific checkers provided an inverse output solution that yielded a checkerboard pattern that gave not only indicative resolution length information but also information on the direction dependence of the resolution.

**Key words:** resolution matrix, spatial or temporal resolution, resolution length, inverse problem, linear or linearized inversion
1. Introduction

A class of geophysical problems involves retrieving the covered structure \( \mathbf{m} \) (an \( m \)-vector including: \( m_1, m_2, \ldots, m_m \)) from an outside observation \( \mathbf{d} \) (a \( n \)-vector) on the basis of a forward equation \( \mathbf{d} = g(\mathbf{m}) \). This is a typical inverse problem that yields a solution \( \hat{\mathbf{m}} = g^\#(\mathbf{d}) \), where \( \hat{\mathbf{m}} (m_1, m_2, \ldots, m_m) \) is the expected solution vector and \( g^\# \) is the true or generalized inverse of the observation operator \( g \). If the observation error \( \delta \mathbf{d} \) is given, the solution uncertainty \( \delta \mathbf{m} \) can also be obtained. The determination or inversion of \( \mathbf{m} \) has been a significant focus of mathematical inverse studies for quite some time. Textbooks are readily available (e.g., Menke, 1984; Tarantola, 1987; Parker, 1994; Tarantola, 2004; Aster et al., 2005), discussing methods for determining the solution efficiently or at least acceptably. For example, for a very large and sparse linear/linearized equation, a solution may be obtained quickly by LSQR (Paige and Saunders, 1982a; b), which has become a standard algorithm for solving tomographic problems (Zhang and Thurber, 2007); even global solutions to complex nonlinear problems may be efficiently obtained via optimization methods (e.g., Goldberg et al., 1992; Sambridge, 1999; An and Assumpção, 2004; Lawrence and Wiens, 2004).

The determination of both \( \mathbf{m} \) and \( \delta \mathbf{m} \) are insufficient for a real geophysical study. In geophysics, each parameter (e.g., \( m_i \)) of a model \( \mathbf{m} \) represents information at a certain spatial or temporal position, expressed as \( m_i(x, y, z, t) \). The spatial and temporal resolutions (or the resolution length), that is, the size of the smallest possible feature that can be detected, as a function of the position \((x, y, z, t)\) are relevant quantities and are informative for appraising the solution \( \mathbf{m} \) as well as the solution uncertainty \( \delta \mathbf{m} \). The computation of solution’s spatial resolution is nontrivial, and is more difficult than
solving an inverse problem. Most geophysical studies, except for tomographic studies, almost uniformly neglect the calculation of a practical spatial resolution.

A generally preferable spatial resolution estimate, widely used in seismic tomography studies, involves visual inspection of the restoration of a synthetic structure (e.g., checkerboard tests) (Lévêque et al., 1993; Thurber and Ritsema, 2009; Feng and An, 2010). The average resolution length in such cases is taken to be equal to half of the recovered checker dimension (Lebedev and Nolet, 2003). The observation of small-scale heterogeneities in a visual inspection assessment is too easily mistaken as an indicator of a high resolution, and the creation of synthetic structures required by the tests becomes difficult for models with complex discretization.

An effective strategy for obtaining the model resolution length scale is the use of Backus–Gilbert resolution kernels (Backus and Gilbert, 1968), also referred to as a resolution matrix. Backus–Gilbert kernels can indicate the finiteness of a model parameterization, the incompleteness of data coverage, and the effects of damping applied during the inversion (Ritsema et al., 2004). The kernels have a finite spatial extent (Ritsema et al., 2004) corresponding to the resolution length. The resolution matrix can be estimated easily for a moderate inversion (Berryman, 2000a; b), and this is also feasible for large-scale problems through Lanczos iteration inversion (Jackson, 1972; Yao et al., 1999; Zhang and Thurber, 2007), the truncated SVD inversion (Aster et al., 2005; Kalscheuer and Pedersen, 2007), the Choleskey factorization inversion (Boschi, 2003; Soldati and Boschi, 2005), or the resolution-matrix approximation (Nolet et al., 1999; Vasco et al., 2003).
Very large inverse problems can be solved using efficient methods, e.g., LSQR (Paige and Saunders, 1982b); however, estimating the resolution matrix of a large problem is nontrivial, and not all resolution matrices can provide resolution length information. The Backus–Gilbert resolution kernels treat essentially underdetermined linear problems (Tarantola, 2004), and every estimate requires a full inversion of the system (Nolet, 2008). A formal resolution test, therefore, becomes impractical for extremely large problems (Thurber and Ritsema, 2009). In reality, the main utility of a resolution matrix is to measure the resolution obtainable from a data set according to the degree to which the resolution matrix approximates the identity matrix (Jackson, 1972; Berryman, 2000b). If the resolution matrix is an identity matrix, the solution $m$ can be taken as unique, each element is perfectly resolved (Jackson, 1972; Tarantola, 2004), and the resolution length estimated from the resolution matrix should only be determined by the parameter’s spatial extent.

Several researchers have published other skills to calculate the spatial resolution in typical inversions or tomographic studies. Derivative weighted sums measure the sampling of each node; they have long been used as an approximate measure of resolution (Thurber, 1983; Toomey and Foulger, 1989; Thurber and Eberhart-Phillips, 1999; Zhang and Thurber, 2007). Yanovskaya & Kozhevnikov (2003) estimated the radius of the averaging area in surface-wave tomography to evaluate the resolution of the associated data. Fichtner & Trampert (2011) explained how to retrieve resolution information from the Hessian operator of an inverse problem.
Despite being crucial for solution appraisal, memory-efficiency estimates of the spatial resolution of a general inversion have yet to be developed. Here, a simple method is presented for inverting a solution’s spatial/temporal resolution on the basis of a limited number of forward and inversion calculations.

2. Traditional resolution matrices and resolution length

Spatial resolution calculations often arise from an analysis of a resolution matrix (Nolet, 2008; Thurber and Ritsema, 2009). Resolution matrices described in previous publications can be divided into three classes, as introduced below.

Consider a linear inverse problem in which the forward and inverse equations can be expressed, respectively, by equations (1) and (2):

\[
d = Gm, \quad (1)
\]

\[
m = G^*d, \quad (2)
\]

where \( G^* \) is a true or generalized inverse of an \( n \)-by-\( m \) observation matrix \( G \). Replacing \( d \) in equation (2) by the expression in the forward equation (1) yields (Jackson, 1972; Menke, 1989):

\[
m = Rm, \quad (3)
\]

and

\[
R = G^*G, \quad (4)
\]

where \( R \) is an \( m \)-by-\( m \) matrix referred to as the model resolution matrix, resolution operator, or resolving kernel. Here, \( R \) is referred to as the direct resolution matrix. Equation (3), which embodies the basic assumption underlying the Backus–Gilbert inversion method (Backus and Gilbert, 1968), defines a linear projection of which
solution \( m \) can be a local average calculated by integrating the model \( m \). If \( G^\tau \) is not calculated, SVD (singular value decomposition) analyses gives the solution (Jackson, 1972):

\[
R = VV^T, \tag{5}
\]

where \( V \) is unitary.

The main utility of a resolution matrix is to provide a measure of the resolution obtainable from the data, and this measure is based on the degree to which the resolution matrix approximates the identity matrix (Jackson, 1972; Berryman, 2000b). If \( G \) is not a full rank matrix, \( R \) will not be an identity matrix, which indicates that some parameter (e.g., \( m_i \)) in the model \( m \) cannot be constrained well by the observation. The inverse problem under such conditions is ill-posed. Figure 2a,b shows the resolution matrix \( R \) calculated by SVD for a simple underdetermined inverse problem, shown in Figure 1. The inverse problem gives a null value for parameters in the range 86–100. These parameters are totally unresolved. If the observation matrix \( G \) is a full rank matrix, \( R \) should be an identity matrix because it is the product of a full rank matrix \( G \) and its inverse. An identity resolution matrix such as that shown in Figure 2c,d indicates that the model was recovered exactly and that the inversion was perfect (Jackson, 1972; Tarantola, 2004).

If a resolution matrix is a nearly diagonal matrix, each element (e.g., \( m_j \)) is in fact the weighted sum of nearby values \( m_j \), where \( m_j \) are near \( m_i \) based on equation (3) (Jackson, 1972). The resolution matrix can, therefore, provide some information about the resolution length. For a full rank observation matrix \( G \), the identity matrix \( R \) (e.g., in Figure 2c,d) indicates that the resolution length at a position \( m_j \) only corresponds to the
spatial extent of \( m \). Considering that the half width of the recovered checker extent can be taken as the resolution length (Lebedev and Nolet, 2003), the half spatial extent of a parameter, \( m \), can also be taken as the resolution length of the parameter, given an identity resolution matrix. For the example in Figure 2a,b, the shaded squares indicate that the parameters inside each square are constrained by a single observation and should have the same inverted values. The resolution lengths of the parameters located inside each square are half of the square width and they are marked in Figure 2b by horizontal red bars centered on the diagonal elements of the matrix.

For an ill-posed problem, such as the underdetermined problem shown in Figure 1, regularization (e.g., damping, smoothness/flatness, or other a priori constraints, here represented by the \( k \)-vector \( c \) and by the \( k \)-by-\( m \) matrix \( C \)) must be used to stabilize the inversion. The forward equation then assumes a form similar to equation (6):

\[
b = Am,
\]

and

\[
b = \begin{bmatrix} d \\ \lambda e \end{bmatrix}, \quad A = \begin{bmatrix} G \\ \lambda C \end{bmatrix},
\]

where \( A \) is an \((n+k)\)-by-\( m \) matrix, and \( \lambda \) is the weight required to balance the a priori constraints and observations in the inversion. The solution to equation (6) can be expressed by

\[
m = A^{-\frac{\lambda}{\lambda}} b.
\]

In an approach similar to that used to obtain the resolution matrix \( R \) in equation (4), the forward and inverse equations (6) and (8) yield a new resolution matrix \( R \) (referred to here as a regularized resolution matrix),

\[
R = A^\frac{\lambda}{\lambda} A.
\]
Many inversion applications use existing inversion programs directly. For example, iterative inversion systems such as LSQR (Paige and Saunders, 1982a; b) are often directly used in tomographic studies. Because inversion programs only provide limited options to include a priori constraints (e.g., in LSQR only damping is provided), if one wants to use one’s preferred constraints, the simplest way is to take the regularized vector \( \mathbf{b} \) and matrix \( \mathbf{A} \), as expressed in Equation (6), as the input observation vector and observation matrix in the inversion programs, respectively. Subsequently the resolution matrices provided by the programs are examples of a regularized resolution matrix \( \mathbf{R} \).

Basically, stabilization on the equation (6) requires, at a minimum, that \( \mathbf{A} \) is a full rank matrix. \( \mathbf{R} \) should then be an \( m \)-by-\( m \) identity matrix (e.g., in Figure 2c) because \( \mathbf{R} \) is the product of the full rank \( \mathbf{A} \) and its inverse. The identity resolution matrix \( \mathbf{R} \) (e.g., in Figure 2c) indicates that the resolution length of each parameter is equal to half the spatial extent of the parameter; however, the resolution matrix \( \mathbf{R} \) from the original forward and inverse equations indicated that some model parameters could not be well solved, and none of the parameters (Figure 2a) have a resolution length equal to half their spatial extent. The resolution matrix \( \mathbf{R} \) cannot, therefore, provide any resolution length information. On the other hand, if we still use the original observation vector \( \mathbf{d} \) and matrix \( \mathbf{G} \) in an inversion program without any constraints, the resulting resolution matrix \( \mathbf{R} \) is meaningful, but the inversion itself is ill-posed.

The vector \( \mathbf{c} \) is often an empty vector, the components of which are zero. In this case, \( \mathbf{c} \) can be ignored, and the vector \( \mathbf{b} \) in the inverse equation (8) can be replaced by \( \mathbf{d} \). In this case, the solution to equation (6) will be identical to the solution to equation (10),

\[
\| \mathbf{Gm} - \mathbf{d} \|^2 + \| \mathbf{Cm} \|^2 = \min . \tag{10}
\]
If C is the identity matrix, equation (10) assumes the form of typical Tikhonov regularizations. Furthermore a third class of resolution matrix (R) may be obtained based on the forward equation (1) and inverse equation (8),

$$R = A^*G.$$  

(11)

Here, R is referred to as a hybrid resolution matrix because it combines the original observation matrix G and another matrix, the regularized observation matrix A. The resolution matrices in many approaches (e.g., Crosson, 1976), especially approaches that offer spatial resolution information (e.g., Barmin et al., 2001; Boschi, 2003; Soldati and Boschi, 2005), belong to this class. Obviously, even for an over-determined problem, the hybrid resolution matrix R is seldom an identity matrix because it is the product of the observation matrix G and the inverse of another matrix A, and the difference between A and G often cannot be ignored. By the same reasoning, R can provide information about the original observation matrix G, the a priori matrix C, and the balance weight λ. The hybrid resolution matrix R (Figure 2e,d) for the underdetermined problem shown in Figure 1 can be very similar to R (Figure 2a,b).

Consider an over-determined (n > m) problem based on the problem presented in Figure 1. A new observation matrix \(G_o\) is defined as the combination of the above G and the m-by-m identity matrix I,

$$G_o = \begin{bmatrix} G \\ I \end{bmatrix}. \quad (12)$$

Using this new observation matrix \(G_o\), the direct resolution matrix R is certainly an identity matrix (like Figure 2c) because \(G_o\) is a full rank matrix. Then each solution parameter can be exactly determined, even without a priori constraint. The resolution length at a position at which a parameter located is equal to half of the spatial extent of
the parameter. \( R \) should also be equal to the identity matrix; however, the identity of \( R \) with the identity matrix depends on the weight \( \lambda \). For a small weight \( \lambda \), the spatial resolution indicated by \( R \) in Figure 3a is similar to that expected from the identity matrix. For a larger \( \lambda \), the hybrid resolution matrix (in Figure 3b) ironically becomes very similar to the resolution matrix of an underdetermined problem, such as that presented in Figure 2e. This similarity mistakenly indicates that the resolution length is much larger than the real resolution length. Therefore, a poor value of \( \lambda \) may lead to an incorrect estimate of the resolution length based on a hybrid resolution matrix \( R \).

The above discussion suggests that a direct resolution matrix \( R \) can provide resolution length information and the perfectness of the inversion; however, this information is seldom pursued in studies because the inversion is often ill-posed and must be regularized. The second class of resolution matrices, \( R \), can evaluate the inversion perfectness for a given mathematic system, but such matrices cannot provide resolution length information. The third class resolution matrices, \( R \), can be used to extract resolution length information and to evaluate the perfectness of an inversion; however, these matrices depend on the regularization contribution to the inversion. From this perspective, it may be optimal to simultaneously calculate all three resolution matrices in a given study, although it is often impossible to perform the required matrix operations in the context of very large inverse problems. Below, I will introduce a new class of resolution matrices that are suited for very large inverse problems and do not need matrix operation.
3. Statistical resolution matrix and resolution length

Basic equations for a linear problem

Unlike previous treatments, the goal here is to directly invert for a resolution matrix based on the linear projection equation (3). This treatment is distinct from that applied in the previous three classes of resolution matrices (R, R, and R). Here, the inverted resolution matrix is represented by the symbol, R, and the projection definition from the real model to a calculated solution is expressed as equation (13),

\[ \mathbf{m} = \mathbf{Rm}. \] (13)

Using equation (13), any solution parameter \( m_j \) can be expressed as:

\[ m_i = \sum_{j=1}^{m} r_{ij} m_j \] (14),

where \( r_{ij} \) is the \( i \)th row and \( j \)th column element of \( \mathbf{R} \). Each row (e.g., \( i \)th row) of \( \mathbf{R} \) is a resolution map that defines the contributions of all model parameters to the \( i \)th solution parameter. Alternatively, equation (14) indicates that each solution parameter is generally derived from the weighted averages of the neighboring solution elements/parameters for a near-diagonal resolution matrix (Jackson, 1972). A small \( r_{ij} \), indicates that the true model parameter \( m_j \) contributes little to the solution parameter \( m_i \).

A long resolution length indicates that neighboring model elements/parameter \( (m_j) \) at a long distance from \( m_i \) can contribute significantly to the parameter \( m_j \). In this case, \( r_{ij} \) should be small according to equation (14) or, say, \( m_i \) contributes little to \( m_j \). This implies that the amplitude uncertainty \( (\delta m_i) \) of the solution parameter \( m_i \) may be large.
In general, a long resolution length may correspond to a large uncertainty in the anomaly amplitude, and vice versa.

The sum of the weights can be a constant, \( c_i \), as expressed by:

\[
\sum_{j=1}^{m} r_{i,j} = c_i \quad (15)
\]

If the summation \( c_i \) is equal to 1, the solution parameter \( m_i \) can be considered as unbiased under the condition that equations (13) or (14) yield a true average (Nolet, 2008). Otherwise, if \( c_i \neq 1 \), \( \sum_{j=1}^{m} r_{i,j} m_j \) in equation (14) is not the true average over the model parameters (Nolet, 2008). In this case, the weighted average of the solution elements/parameters should be written as:

\[
m_i = \frac{\sum_{j=1}^{m} r_{i,j} m_j}{\sum_{j=1}^{m} r_{i,j}} \quad (16)
\]

**Direction-independent or 1D resolution matrix estimation**

For a good inversion, the resolution matrix must be an identity matrix or nearly diagonal (Jackson, 1972), i.e., a given parameter (e.g., \( m_i \)) and its neighbors can provide a large contribution/weight to the average summation for a solution parameter (\( m_i \)), and the weights \( (r_{i,j}, j = 1,m) \) should decrease rapidly with the distance from \( m_i \) to another parameter (\( m_j \)); therefore, each row of the resolution matrix \( (r_{i,j}, j = 1,m) \) can be approximated by either a cone-shaped function (Barmin et al., 2001) or a Gaussian function (e.g., Nolet, 2008). Here, the approximation of a Gaussian function is used. In
one dimension, the approximate values of $r_{i,j}$, the $i$th row and $j$th column of $\mathcal{R}$, in the Gaussian function can be written as

$$r_{i,j} = a_i e^{-\frac{(x_i-x_j)^2}{2\sigma_i^2}} = a_j e^{-\frac{L_{i,j}^2}{2\sigma_i^2}} = e^{-\frac{L_{i,j}^2}{2(w_i/1.17)^2}},$$

(17)

where $x_i$ and $x_j$ represent dimension (e.g., $x$, $y$, $z$, or $t$) positions for $m_i$ and $m_j$, $L_{i,j}$ is the Euclidean distance between the dimensional positions of the parameters $m_i$ and $m_j$, $a_i$ is a constant, $\sigma_i$ represents the half width of the peak at $\sim$60% of its full height, and $w_i (= 1.17\sigma_i)$ is the half width at half maximum (and is often used to represent the shape's width). The spatial form (curve or surface) of a Gaussian function is very similar to the form of a recovered checker in a traditional checkerboard test. Since the half width of the recovered checker dimension can be taken as the resolution length (Lebedev and Nolet, 2003), half of the Gaussian width, $w_i$, can be taken as the resolution length at the position $m_i$; therefore, $w_i$ will be referred to as the resolution length at position $i$. If $a_i$ is not considered, a normalized inverted resolution matrix $\tilde{\mathcal{R}}$ can be constructed according to

$$\tilde{r}_{i,j} = e^{-\frac{L_{i,j}^2}{2\sigma_i^2}}. \quad (18)$$

Replacing $r_{i,j}$ in equation (16) by the expression given in equation (17) yields

$$a_i \sum_{j=1}^{m} e^{-\frac{L_{i,j}^2}{2\sigma_i^2}} m_j = a_j \sum_{i=1}^{m} e^{-\frac{L_{i,j}^2}{2\sigma_i^2}} m_j = e^{-\frac{L_{i,j}^2}{2(w_i/1.17)^2}}.$$ \quad (19)

If we write
\[
f(w_i, m) = \frac{\sum_{j=1}^{m} e^{-\frac{(l_{i,j} - m_j)^2}{2}\sigma_i^2}}{\sum_{j=1}^{m} e^{-\frac{(l_{i,j} - m_j)^2}{2}\sigma_i^2}} = \frac{\sum_{j=1}^{m} e^{-\frac{(l_{i,j} - m_j)^2}{2(\sigma_i^2/1.17)^2}}}{\sum_{j=1}^{m} e^{-\frac{(l_{i,j} - m_j)^2}{2(\sigma_i^2/1.17)^2}}},
\]

we obtain the equation

\[
m_j = f(w_i, m).
\]  

Because a resolution matrix does not depend on the specific model or observation values, but rather on the exclusively properties of the observation matrix \( G \), we can use a random synthetic model to solve equation (21). Given a random synthetic model \( m^1 \), we can calculate the synthetic observation from \( m^1 \) using equation (1). We can then obtain a solution \( m^1 \) directly using the inverse equation (8). For a very large inverse problem, the forward and inversion calculations (e.g., inversion by LSQR) are often much easier to obtain than the traditional resolution matrices. If the model and solution given in equation (21) are known, \( w_i \) becomes a unique unknown in equation (21). In this case, equation (21) can be solved; however, the solution \( w_i \) may depend on the given \( m^1 \) and \( m^1 \). For \( ns (> 1) \) given different random synthetic models \( M (= m^1, m^2, \ldots, m^{ns}) \), we can obtain \( ns \) respective solutions \( M (= m^1, m^2, \ldots, m^{ns}) \), and we can construct \( ns \) nonlinear equations similar to equation (21). \( w_i \) is the only unknown in the equations and can be obtained by minimizing equation (22),

\[
\sum_{j=1}^{ns} |m'_j - f(w_i, m')| = \min
\]

In most real inverse problem, \( w_i \) has a limited number of possible values. For example, in a 1D inversion for an equal thickness layered model (e.g., see Figure 1) with \( m \) parameters and thickness \( (h) \), \( w_i \) can be only one of \( 0.5h-mh \). For the unique unknown
equation (22), an exhausted grid search is the most powerful and easiest approach to obtaining the resolution \( w_i \). As \( ns \) increases, the dependence of \( w_i \) on the given synthetic model decreases, and the final solution \( w_i \) can be obtained by increasing \( ns \) until \( w_i \) becomes stable. By repeating the above grid-search inversion at parameter positions other than \( i \), we can obtain all resolution lengths, \( w_i (i = 1, m) \). The full procedure outlined above is indicatively illustrated by the flowchart in Figure 4.

After obtaining all \( w_i (i = 1, m) \), we can directly construct a normalized resolution matrix \( \tilde{\mathbf{R}} \) using equation (18). Note that we can obtain \( a_i \) using equations (14) and (17), and we can then construct the inverted resolution matrix \( \mathbf{R} \) using equation (17); however, because we mainly want to extract the resolution length from a resolution matrix, which \( w_i \) can provide directly, \( \tilde{\mathbf{R}} \) is mainly shown in the figures.

**Method validity**

The linear projection (equations (3) and (13)) from the real model to the calculated solution to a linear inverse problem is well accepted and widely applied in linear inverse studies. In contrast with the traditional resolution matrices obtained by operating on an observation matrix, the resolution matrix presented here may be solved directly using random synthetic models and their respective solutions. The solution to each synthetic model may be obtained by forward and inversion processes on the basis of a real observation matrix and other parameters (e.g., regularization), similar to the process used to invert real data. The solution is influenced by all parameters and methods used, from the forward calculation to the final inversion; however, the traditional resolution matrices are determined only using the observation matrix and a regularization procedure; therefore, the inverted resolution matrix presented here is not identical to the
The above approach was used to invert for the statistical resolution matrix (see Figure 5) of the linear inverse problem presented in Figure 1. Twenty-five ($ns = 25$) pairs of random synthetic models and solutions were used to invert the statistical resolution matrix. Figure 5b showed that the resolution lengths at the positions for the parameters $m_i(i = 1–85)$ were similar to those calculated directly from equation (4). These values were reasonable at the positions for the parameters $m_i(i = 86–100)$. Because the flatness constraint is associated with the parameters $m_i(i = 86–100)$, similar to the neighbors $m_i(i = 81–85)$, the resolution lengths for the positions of the parameters $m_i(i = 86–100)$ should be 10 km (the half width of the spatial extent, 20 km) similar to the resolution lengths determined using the inversion approach of the present study, as shown in Figure 5b.
During the extraction of resolution length information from a resolution matrix, it is assumed that the diagonal components of each row should be largest, and that the values of the other components should decrease as the distance from the respective component to the diagonal component increases, as illustrated by the cone-shaped function (Barmin et al., 2001) or the Gaussian function used here, and also by Nolet (2008). However, the real resolution matrix does not always behave in this way. The geometry of the resolution matrices shown in Figure 2a,e is neither symmetric nor Gaussian and differs from the above geometry variation assumption. Ritzwoller et al. (2001) described a Gaussian smoothness constraint on the solution parameters associated with a surface-wave tomography inversion and estimated the resolution lengths. This constraint forces the solution parameters to satisfy an equation similar to \( \mathbf{m} = \mathbf{Cm} \), but the constraint does not directly influence the linear projection equation, \( \mathbf{m} = \mathbf{Rm} \), which was the assumption used to estimate the resolution length.

In general, the resolution length estimated from a resolution matrix based on a specific geometry shape can often provide an approximate indicator of the spatial resolution. The similarity between the resolution lengths from, respectively, \( \mathbf{R} \) and \( \mathbf{R} \), as shown in Figure 5b, indicates that the approximation can adequately represent the real resolution length for a linear inverse problem, at least for the inverse problem in Figure 1.

*The flatness influences the inverted resolution length*

The influence of flatness was tested by estimating the statistical resolution matrices using different weights (\( \lambda \)) in equation (7), which balanced the flatness constraint and the synthetic observations. Here, the forward equation (6) was inverted using the LSQR which is a conjugate gradient method. For \( \lambda = 1 \), the results shown in Figure 6a,b were
identical to those shown in Figure 5. Figure 6 showed that the resolution lengths increased with the weights ($\lambda$) on the flatness values, as expected. In general, a large weight indicates a large contribution from the flatness constraint in the inversion and results in a long resolution length.

Nonlinear case

For a small model space around a real model (or, for example, for a given reference model $m_0$ sufficiently close to a real model, $m$) the nonlinear forward equation, $d = g(m)$, can be approximated using a linearization procedure,

$$\Delta d = G \Delta m,$$

(23)

where $\Delta d = d - g(m_0)$ and $\Delta m = m - m_0$. The equation (23) is similar to the equation (1). Similarly, the statistical resolution matrix was defined as

$$\Delta m = R \Delta m.$$  

(24)

Using the above equation, an approximate statistical resolution matrix for the nonlinear problem could easily be determined based on the approach described above for estimating the statistical resolution matrix of a linear problem.

On the basis of equation (24) we obtain:

$$m = \Delta m + m_0 = R \Delta m + m_0.$$  

(25)

In an inversion iteration for a linearized nonlinear inversion, the inverse problem is, in fact, a linear inverse problem based on the relevant reference model. If the linear inversion during the iteration is perfect, the resolution matrix $R$ can be the identity matrix $I$ or, say, $R = I$. In this case we obtain:

$$m = R \Delta m + m_0 = R(\Delta m + m_0) = R m.$$  

(26)
Equation (24) is similar to equation (13). However, the resolution matrices defined by these two equations have different levels of significance. The resolution matrix defined in equation (24) does not represent the relation between the model and the solution like that in equation (13), but it is an approximate relation between the model difference and solution difference for a given reference model. Therefore, we write $\mathcal{R}$ in equation (24) as $\mathcal{R}(m_0)$. Equations (23) to (26) indicate—but only when the given reference model is close enough to the true model and the linear inversion for the iteration is perfect—that the resolution matrix defined in equation (24) can have a similar significance as that in equation (13). However, the above conditions are rarely true for nonlinear inversion. Therefore, the resolution length estimated from $\mathcal{R}(m_0)$ is often not our expected absolute model resolution length for an inversion, but an approximate resolution length of the model difference related to a given reference model. The real model resolution length will depend on both the reference model and the nonlinearity of observation operator $g$ of the inverse problem. In this case, the primary use of the resolution length retrieved from $\mathcal{R}(m_0)$ is to measure the relative solution precision between parameters based on the degree to which the row of the matrix approximates a delta function, on the basis of the given reference model, as indicated by the example below.

One-dimensional S-velocity inversions from surface wave dispersions are typical seismological nonlinear inverse problems widely applied in crust and lithosphere studies. Although the hybrid resolution matrix may be applicable to linearized inversions (Herrmann and Ammon, 2002), no publications have explicitly described the spatial resolution in related studies. Here, a synthetic example is presented for
calculating the matrices $\mathbf{R}$ and $\mathcal{R}(\mathbf{m}_0)$ for a 1D inversion. The spatial resolution is also estimated: see Figure 7. The program surf96 (Herrmann and Ammon, 2002) was used to performed the forward calculation and linearized inverse iterations, and outputs the hybrid resolution matrix ($\mathbf{R}$) at each inverse iteration. The vertical S-velocity model included 39 layers (each 5 km thick) and a half-space layer at the bottom (Figure 7a), and damping in all inversions was 0.01. The value of $\mathbf{R}$ at the 15th iteration is shown in Figure 7b. The last column of the hybrid resolution matrix is related to the bottom half space layer, and each component in the column differs significantly from the other components in the row. I discarded all matrix components related to the bottom layer in the normalized hybrid resolution matrix (Figure 7c). On the basis of the output model during the 15th iteration, I made 30 random synthetic models and their respective inverted solutions using the program, finally the statistical resolution matrix (Figure 7d) was obtained by the method suggested above.

As shown in the comparison in Figure 7c,d, the spatial resolution lengths from the matrices $\mathbf{R}$ and $\mathcal{R}(\mathbf{m}_0)$ generally increased with the depth. For example, the resolution lengths of $\mathbf{R}$ increased from 12 km (the spatial extent of a parameter is equal to the layer thickness, 5 km) for the 10th parameter to 25 km for the 20th parameter. The resolution lengths for $\mathcal{R}(\mathbf{m}_0)$ increased from 8 km to 20 km. Non-negligible differences between the two resolution matrices were observed. Aside from the difference in the resolution length values of $\mathbf{R}$ and $\mathcal{R}(\mathbf{m}_0)$, the resolution lengths of the layers at or near the model bottom (Figure 7d) were very small (<10 km), which differed from the conditions of the hybrid resolution matrix (Figure 7c). Because the statistical resolution matrix could be obtained directly using synthetic models and their respective solutions, the small resolution length at or near the bottom provided special information about the solutions...
to the random synthetic models. In fact, a high spatial resolution for a parameter may indicate that the parameter can be easily and well inverted. Figure 7a shows that the parameters at or near the bottom layer could approach the synthetic model (black line) after only 3 iterations in this synthetic example, indicating a high resolution in the layer at or near the bottom. The small resolution length calculated based on the statistic resolution matrix should, therefore, be reasonable.

High-dimension or direction-dependent cases

For \( nd \) dimensions \((nd > 1)\), a Gaussian-shaped function for an element \( r_{i,j} \) at the \( i \)th row and \( j \)th column of \( \mathbf{R} \) can be written as

\[
 r_{i,j} = a_i e^{-\sum_{k=1}^{nd} \frac{(x_i^k-x_j^k)^2}{2(\sigma_i^k/\sqrt{1.17})^2}} = a_i e^{-\sum_{k=1}^{nd} \frac{L_{i,j}^k}{2(\sigma_i^k/\sqrt{1.17})^2}} .
\]

A resolution matrix for a specific direction can be expressed similar to the equation given. For a resolution matrix that does not depend on the dimension or direction, all \( w_i^k \) \((k = 1, nd)\) are equal and can be written as \( w_i \). Therefore, the statistical resolution matrix can be inverted using the same approach as was described for the 1D case, above. Otherwise, a resolution matrix may be inverted along each dimension separately using the equation with one unknown. All \( w_i^k \) \((k = 1, nd)\) also can be inverted simultaneously. Using the same approach described for equations (19)–(22), we can obtain \( nd \)-dimensional equations according to

\[
 m_i = \frac{\sum_{j=1}^{m} a_i \left( \sum_{k=1}^{nd} e^{-\frac{L_{i,j}^k}{2(\sigma_i^k/\sqrt{1.17})^2}} \right) m_j}{\sum_{j=1}^{m} \left( \sum_{k=1}^{nd} e^{-\frac{L_{i,j}^k}{2(\sigma_i^k/\sqrt{1.17})^2}} \right)} = \frac{\sum_{j=1}^{m} m_j}{\sum_{j=1}^{m} \sum_{k=1}^{nd} e^{-\frac{L_{i,j}^k}{2(\sigma_i^k/\sqrt{1.17})^2}}} = \sum_{k=1}^{nd} f(w_i^k, \mathbf{m})
\]
We can then obtain equation (29), which includes \( nd \) unknowns, \( w_i^k \) \((k = 1, nd)\). The approach used to solve equation (22) was also used to obtain the unknowns by minimizing equation (29),

\[
\sum_{i=1}^{ns} |m_i - \sum_{k=1}^{nd} f(w_i^k, m)| = \min
\]  

(29)

4. Discussion

Required synthetic model number

The number of synthetic models determines the computational efficiency of the above approach. Each model and its respective solution are analogous to a measurement sample for resolution length, \( w_i \). Consequently, the determination of the minimum number of synthetic models required to provide a stable inversion of \( w_i \) or a resolution matrix can be considered as a statistical problem of minimum sample size determination on the basis of a given confidence interval. Because a measurement is only taken for one parameter \( (w_i) \), the minimum sample number \( (ns) \) will not depend on the total number of parameters in the model but instead on the number of all possible values of \( w_i \). As pointed out above, with increasing \( ns \), the mean effect of all samples can result in a stable \( w_i \) that is close to the true resolution length. Therefore, this problem can be taken as satisfying the central limit theorem.

The central limit theorem states that when the sample size is large, sample means will fall within \( \pm Z_{\alpha/2} \) times the standard errors of the population mean, that is,

\[
\bar{w}_i - E_i < w_i < \bar{w}_i + E_i,
\]

(30)

and
\[ E_i = Z_{\alpha/2} \left(\frac{d_i}{\sqrt{\text{ns}}}\right) \]  

(31)

where \( E_i \) is called the margin of error, \( Z_{\alpha/2} \) is a constant for the relevant confidence level, and \( d_i \) is the standard deviation of all possible \( w_i \). If the margin of error is known, the minimum sample size for the given confidence level can be obtained as:

\[ \text{ns} = \left(\frac{Z_{\alpha/2} d_i}{E_i}\right)^2 \]  

(32)

For the 1D example in Figure 1, the real \( w_i \) can only be in the range of 0–20h. If all possible values of \( w_i \) have the same probability, the standard errors of the values, \( d_i \), should be \(-5.8h\). If the 95% confidence level is used, \( Z_{0.025} = 1.96 \). If we assume a margin of error \( (E_i) \) of 2.5h, we obtain \( \text{ns} = 20 \). As described above, a real test showed that 25 synthetic models \( (\text{ns} = 25) \) were sufficient to obtain a stable \( w_i \).

The minimum sample number was also tested for by preparing a large inverse example. Here, a 2D Rayleigh wave group-velocity tomographic inversion was performed on the Antarctic. The inverted results using real observations of the Antarctic will not be discussed here, rather, the observation matrix will be used as an example for extracting the resolution length information. The inverse horizontal 2D model was parameterized using 12163 equal-area hexagonal and pentagonal cells with widths of \(-1\) great-circle degree (see Figure 8a). The application of traditional checkerboard tests on the tetragonal cells was impossible due to the use of hexagonal and pentagonal cells. The inversion was a linear inverse problem, and LSQR was used here to obtain a solution. For this problem, the possible values of \( w_i \) are in the range of 0–15 degrees, and the standard errors of the values should be \(-4.35\) degrees; if the margin of error is \(-0.5\) degree, the minimum sample number \( (\text{ns}) \) should be \(-290\) at the 95\% confidence level.
according to equation (32). Figure 8b,c presents a random synthetic model and its inverted solution. Statistical resolution matrices were obtained using different $ns$ pairs of random synthetic models and their respective solutions; the resolution lengths from the matrices using 20, 200, and 300 samples are shown in Figure 8d–f. A comparison among Figure 8c,d,f reveals that $ns = 300$, which is similar to the predicted minimum sample number above and sufficient to obtain a stable resolution length for the entire study region. The resolution lengths (Figure 8f) are consistent with the path coverage (Figure 8a). For example, within the continental Antarctic, the path density was high (Figure 8a); the resolution length (Figure 8f) was also high and approached the cell dimensions.

In the 1D tests using 100 parameters, as described above, 25 synthetic models ($ns = 25$) were sufficient to obtain a stable $w_i$. Here, a 2D example with 12163 parameters required only 300 models. The tests showed that fewer than several hundred pairs of synthetic models and solutions produced a stable $w_i$, and the minimum number of synthetic models depends on the possible values of $w_i$ but not on the number of parameters. The minimum sample numbers in the real tests are similar to the prediction by the central limit theorem.

*Does the inversion of a random synthetic model simulate a checkerboard test?*

Interestingly, the solution pattern (Figure 8c) for a random synthetic model (Figure 8b) submitted to the tests described above based on the 2D Rayleigh wave group-velocity tomography of the Antarctic resembled a checkerboard test output. The inverted solution for any random synthetic model was found to display notable patterns similar to the output of a traditional checkerboard test using an input model with checker pattern anomalies. The anomalous pattern size in each solution (Figure 8c) was
comparable to the resolution length scale estimated from the resolution matrix analyses, as shown in Figure 8f.

The checker-like patterns in the solution based on a random synthetic model are also shown using ~4-degree-wide equally sized cells (see Figure 9). Tests (not shown here) on a linearized inversion for a 3D random synthetic model also yielded a checker pattern solution, similar to that shown in Figure 8c and Figure 9c. These tests demonstrated that even though the inverted solutions should depend on the respective random input models, the inverse output solution of any random synthetic input model directly yielded a specific checker pattern that can give some indicative resolution length information.

The resolution length corresponds to the size of the smallest possible feature that can be detected. For a given resolution matrix, the resolution length at the position of a parameter is consistent with the number of neighboring parameters which make an obvious contribution to the solution parameter; see equation (14) and Figure 2b. The inverted solutions (e.g., Figure 8c or Figure 9c) revealed that the resolution length at the position of a parameter (Figure 8f or Figure 9d) is consistent with the number of neighboring parameters which have a value similar to that of the parameter in the inverted solution; in contrast, the parameters in the synthetic/real model (Figure 8b or Figure 9b) are not similar to one another. Put simply, the resolution length estimation can be considered as an analysis of how many neighboring parameters have similar values to each other around a given position. According to this idea, the resolution length could be estimated directly via visualization of the solution inverted from a random synthetic model. Following this line of reasoning, variation in the
model/solution parameterization (e.g., absolute value of the physical quantity, anomaly, and relative anomaly) may influence the inversion for a statistical resolution matrix especially for a linearized nonlinear inverse problem. It seems that the anomaly or relative anomaly could be superior in the model/solution parameterization when retrieving resolution-length information using the approach presented above.

Even though the information obtained from the inverted solution of a random synthetic model depends on said synthetic model, the visualization of the solution inverted from a random synthetic model can not only give some indicative resolution length information (as explained above), but it can also yield the direction dependence of the resolution. Because most of the observation stations (Figure 8a) are located inside continental Antarctica and earthquakes originate at the plate boundaries, the rays have a better defined cross section at a position far from the plate boundary, while rays at positions close to the plate boundary are oriented much more parallel to their neighboring rays. In practice, the resolution length along the parallel direction (nearly parallel to the meridians) of the rays should be longer than that in the normal direction (nearly parallel to the latitude). In the solution (Figure 8c) inverted from a random synthetic model, most of the anomaly pattern looks like a strip with a long axis along the meridian and a short axis along a line of latitude, especially for the oceanic region where the patterns of the anomaly in the input synthetic model (Figure 8b) exhibit little similarity to the strip anomaly in the solution. According to the reasoning above, the strip anomaly pattern in the oceanic region should indicate that the resolution length along the meridian is longer than that along the line of latitude in the oceanic region, which is consistent with indications from the ray distribution.
Resolution lengths of discontinuities between neighboring parameters

Discontinuities or sharp variations between neighboring model parameters (e.g., $m_i$ and $m_j$) provide important information in geophysical studies. Therefore, the resolution lengths of a discontinuity are needed for an inversion, if possible. Using equation (14), an equation was obtained describing the differences between neighboring parameters,

$$\Delta m_{i,j} = m_i - m_j = \sum_{k=1}^{m} (r_{i,k} - r_{j,k}) m_k$$

Equation (33) indicates that the differences between two solution parameters ($m_i$ and $m_j$) depend not only on the entire model, but also on the resolution matrix corresponding to the $i$th and $j$th row elements.

For a perfect inversion, each row of the resolution matrix is a delta function, and equation (33) can be written in terms of equation (34),

$$\Delta m_{i,j} = \sum_{k=1}^{m} (r_{i,k} - r_{j,k}) m_k = m_i - m_j = \Delta m_{i,j}$$

Equation (34), corresponding to a perfect inversion, can only indicate the differences between two neighboring parameters that can be well solved; however, although the width of a discontinuity (such as a Moho) may be infinitely small or thin, the discontinuity cannot be localized to the border between two parameters. In fact, it can only be concluded that the resolution length of a discontinuity does not exceed the sum of the half spatial extents of two parameters, and the real resolution length of a discontinuity can be much shorter. Figure 2a provides an example showing that the discontinuity resolution length at some depth (e.g., 30 or 60 km) can be much higher than the resolution length of a model parameter if the discontinuity is located on a cell’s boundary. A simple way to obtain a reasonable resolution length of a discontinuity in a
perfect inversion is to decrease the parameter spatial extent and re-calculate the resolution matrix until the matrix rows associated with the discontinuity are not delta function. The parameter spatial extent can then be taken as the largest bound on the discontinuity resolution length.

Computation requirements and efficiency

Our approach to calculating the statistical resolution matrix can be divided into two steps: the first step is to create tens or hundreds of random synthetic models and solutions; the second step is to directly invert for the resolution lengths using these synthetic models and solutions. The first step is similar to the procedure used in traditional checkerboard tests; however, the input model is easier to create because it is a random model without a regular anomaly pattern distribution. The forward calculation and the solution inversion procedure applied in the first step are basic calculations for an inversion study. The only coding burden for the skill presented here is to write a short program that inverts the synthetic models and their respective solutions to identify the resolution lengths using a grid search. This coding effort is trivial because it is a one-parameter inversion, and the grid search is the simplest of the inversion methods.

The second step, involving a grid search for the resolution length scale, does not require much memory. The primary memory consumption involves retaining tens or hundreds of pairs of models and their solutions in memory for use in the grid search inversion. The maximal memory required for the computation approaches $2 \times ns \times m$, where $ns$ is generally not more than several hundreds and $m$ is the model parameterization size. The advantages of the approach described here are that the maximal memory consumption is much smaller than that required from a calculation of the resolution matrix using matrix
operations. The idea presented here is especially suitable for very large inverse problems. If the information of the models and solutions are stored on a disk, the memory burden will approach zero, but the computation time will be much longer due to the need for frequent disk reading actions.

The skill presented here is isolated from the specific forward/inversion procedure and does not directly use any relation between observation and model parameters such as observation matrix/operators. Therefore, it can be applied to general inverse problems. For the same reason, one should not be surprised that the skill may have a worse computational efficiency than other skills using the relation between observation and model parameters of the relevant inverse problem. For example, for a small-scale inversion using SVD, the direct/regularized/hybrid resolution matrices can be obtained easily using decomposition matrices from the observation matrix, but calculation of a statistical resolution matrix needs much more time, because $ns$ forward/inversion computations and $m$ 1–unknown grid searches for resolution lengths are needed. On the other hand, since $ns$ is often much smaller than $m$ for large inverse problems, the efficiency of statistical resolution matrix computation is better than that of brute-force calculations of Backus–Gilbert kernels, because the latter need to solve the system $m$ times, and the computation time to solve the system normally is much longer than a 1–unknown grid search for the relevant resolution length.

Lanczos iteration inversions, such as LSQR, can be used to quickly obtain the solution and they have been widely applied to large linear inversions. After some modifications to the inversion codes, a resolution matrix can be obtained (e.g., Yao et al., 1999; Zhang and Thurber, 2007). Because inversion programs only provide limited options to include
a priori constraints, if one wants to use one’s preferred constraints, the simplest way is to take the regularized vector \( b \) and matrix \( A \), as expressed in Equation (6), as the input observation vector and observation matrix, respectively. In this case, the output resolution matrix is a regularized resolution matrix which cannot give any resolution length information, as shown above. Mostly, researchers engaged in tomographic studies have to perform checkerboard tests, which can give indicative resolution length information. The skill presented here can be applied directly to these inversions to obtain resolution lengths. Lanczos iteration inversions are efficient, which will ensure the efficiency of the statistical resolution matrix computation. For the reasons outlined above, statistical resolution matrix computations may be particularly suitable for applications of Lanczos iteration inversions.

A discussion to improve the efficiency of the first step of the forward/inversion computations is beyond the scope of this paper. As for the second step, the efficiency of grid searches for resolution lengths can be easily improved for modern computers equipped with processors containing more than a single core. The 1-parameter grid-search inversion for the relevant resolution length must be repeated for all model parameters. Modern parallel computational approaches (e.g., OpenMP: see www.openmp.org) suitable for repeating calculations, can be easily applied to these codes, which decreases the computation time significantly.

In general, the computation efficiency of the skill presented here may be worse than that of some other skills. However, the skill discussed here can be applied to general problems, and needs little coding work and computation memory. The computation efficiency can be improved using modern parallel computing approaches. Statistical
resolution matrix computation may be particularly suitable for application of Lanczos iteration inversions, because the application can quickly obtain a solution although it is difficult to obtain resolution lengths.

5. Conclusions

Applications of the Backus–Gilbert inversion method are few; however, the basis of the inversion (resolution matrices and kernels that connect real models to calculated solutions) are widely used to evaluate inversion perfectness and to analyze the spatial resolution of a linear inversion. The resolution matrices described in previous publications fall into three classes: (1) direct resolution matrices, which are the product of an observation matrix and its inverse; (2) regularized resolution matrices, which are the product of a regularized observation matrix and its inverse; (3) and hybrid resolution matrices, which are the product of an observation matrix and the inverse of the regularized observation matrix. The first class of matrices provides resolution length information and the perfectness of the inversion but is seldom presented in studies because the inversion is often ill-posed and must be regularized. The second class matrices can be used to evaluate the inversion perfectness under a given mathematical system, but these matrices cannot provide any information about the resolution length. The third class of resolution matrices can be used to extract resolution length information and to evaluate the perfectness of an inversion; however, these matrices depend on the contributions of regularization to the inversion. The simultaneous presentation of all three resolution matrices in a study could fully articulate the inversion perfectness and resolution length information. Unfortunately, it is often impossible to perform any of these calculations in the context of very large inverse problems. Here, a new class of resolution matrices, called the statistical resolution
matrices, is suggested to permit the direct inversion from synthetic models. These matrices are especially suitable for very large linear/linearized inverse problems.

A resolution matrix defines a linear projection in which each solution parameter is derived from the weighted average of the neighboring solution parameters and the resolution matrix elements are the weights. Ideally, a parameter and its neighbors provide a large contribution/weight to the average summation for a parameter in the solution; therefore, all elements in each row of the resolution matrix are assumed to be Gaussian functions over the parameter distances. The resolution matrix exclusively provides the properties of the observation matrix but does not depend on any particular model or observation values; therefore, random synthetic models and their solutions may be used, and the projection equation becomes a nonlinear inversion of only one parameter, the width of the Gaussian shape. The projection can then be directly inverted using a simple grid search method. The inverted widths directly indicate the resolution length.

The tests described here showed that the statistical resolution matrix not only measures the resolution obtainable from the data, it also provides reasonable spatial/temporal resolution or resolution length information. Estimates of the statistical resolution matrix can be used for both direction-dependent and direct-independent resolution estimates. The estimates depend on the real forward/inversion processes and are independent of the approach used to determine the solution inversion. Therefore, it is not necessary to modify the preferred forward/inversion codes/methods if a user wishes to obtain the resolution length. Matrix operations are not needed in the estimation, indicating that
estimating the statistical resolution matrix is especially suitable for very large linear/linearized inverse problems.

Interestingly, the tests showed that for a random synthetic input model without a specific checker pattern, the inverse output solution directly provided a checkerboard anomaly pattern that indicatively provided information on not only the resolution length but also on the direction dependence of the resolution.

All the codes to calculate statistic resolution matrix with examples for the inverse problem in Figure 1 are available through: http://www.seismolab.org/people/meijian/.

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Figure captions

Figure 1. Illustration of a simple 1D ray-propagation 100-parameter model (a) and its observation matrix (b). All model parameters were separated with a distance interval of 1 km. Only 5 observations/rays were used, no observational constraints were imposed on the parameters corresponding to distances of 85–100 km in the observations; therefore, the inverse problem was underdetermined and the observation matrix in (b) was not a full rank matrix.

Figure 2. Three resolution matrices for the underdetermined inverse problem presented in Figure 1. The subfigures b,d,f are the normalized matrices corresponding, respectively, to the subfigures a,c,e. The matrix in (c,d) is an identity matrix. The matrices are calculated by the SVD of (a,b) the $G$ matrix and its inverse, (c,d) the $A$ matrix and its inverse, and (e,f) the $G$ matrix and the inverse of the $A$ matrix. $A$ is a matrix that combines $G$ and the flatness constraints $C$ among all neighboring parameters. The weight ($\lambda = 1$) is used here. In subfigure a), all components of the resolution matrix for parameters, $m_i$ ($i = 86–100$) are 0 because no information is available for the parameters in the observation matrix $G$. The extent of the square-like pattern indicates that the related solution parameters should have the same value in this inversion problem. The extent can be taken as the resolution length of the solution parameters. The horizontal red bars in (d,f) of each model parameter are the same as in (b), and indicate the resolution lengths from (a), but are centralized along the diagonal element of the matrix.
Figure 3. Hybrid resolution matrix calculated by the SVD for an overdetermined inverse problem using the weights (λ) of (a) 1 and (b) 10. The problem’s observation matrix $G_o$ is a matrix that combines $G$ from Figure 1 and the identity matrix. The regularized matrix $A$ combines $G_o$ and $C$.

Figure 4. Flow chart of inversion for the resolution lengths. The procedure can be divided into two steps: (1) generate synthetic models and invert for solutions using your preferred constraints and inversion skill and (2) invert for the resolution lengths.

Figure 5. (a) Statistical resolution matrix, and (b) normalized matrix for the problem presented in Figure 1. The statistical resolution matrix is inverted by the SVD using 25 pairs of synthetic models and solutions. The models were constructed by adding ≤10% random deviations onto a reference constant. In the subfigure (b), the red bars are the same as those shown in Figure 2b. The red contour marks indicate the half height position of the Gaussian shape, which is taken as the resolution length of the statistical resolution matrix.

Figure 6. Statistical resolution matrices (a, c, e) and normalized matrices (b, d, f) of the problem presented in Figure 1, obtained using different weights and the LSQR inversion method. The flatness weights were (a) & (b) 1, (c) & (d) 10, and (e) & (f) 20. The other properties are the same as those described in Figure 5.

Figure 7. Resolution matrices of a Rayleigh wave dispersion inversion for a 1D S-velocity model. The synthetic observation includes the Rayleigh wave group velocities over the periods from 5 to 185 s with intervals of 5 s. The forward calculation and the
linearized inversion was performed using surf96 (Herrmann and Ammon, 2002). The lines in (a) are the synthetic/true model (black) and the reference model or the output model at each linearized inversion iteration, indicated by the colored lines, which vary from red to blue with the iteration number. The blue line indicates the outputted model at the 15th iteration and the red line indicates a constant velocity for all depths in the initial model. (b) The hybrid resolution matrix at the 15th iteration was the output of surf96, and its normalization is shown in (c). The matrix components of the half space were discarded. The resolution matrix in (d) is the normalized statistical resolution matrix based on the above output model (blue line in (a)) at the 15th iteration.

Figure 8. Resolution length information for a ~1-degree-wide cell 2D Rayleigh wave dispersion tomography set over a period of 50 s. Subfigure (a) shows the cells and the 5384 rays from source (circles) to station (triangles) used in this example. Most of the sources are earthquakes, and a few are seismic stations for ambient-noise cross-correlation rays. Subfigure (b) shows one random synthetic 2D model, (c) shows the inverted solution to the synthetic model in (b); (d)–(f) show the spatial-resolution maps from 20, 100, and 300 pairs of random synthetic models and their solutions.

Figure 9. Resolution length information for a ~4-degree-width cell 2D Rayleigh wave dispersion tomography set for a period of 50 s. Subfigure (a) shows the cells and the 5393 rays used in this example. The rays are nearly the same as those in Figure 8a. Subfigure (b) shows one random synthetic 2D model, (c) shows the inverted solution to the synthetic model (b); (d) shows the resolution length map for 300 pairs of random synthetic models and their solutions.
Figure 1

a) Parameter index

Distance (km)

Ray1 → Ray2 → Ray3 → Ray4 → Ray5

Observation index

b) Observation matrix G

Parameter index

Observation index

-1 0 1

Figure 1
Figure 2
Figure 3

a) Resolution matrix
Parameter index (i)
Parameter index (j)
Resolution matrix
-0.68 0.00 0.68

b) Resolution matrix
Parameter index (i)
Parameter index (j)
Resolution matrix
-0.14 0.00 0.14

\( \lambda = 1 \)

\( \lambda = 10 \)
Step 1

- Synthetic observation
- A priori constraints
- Forward calculation
- Inversion
- Random model $m^1$
- Inverted solution $m^1$

1st pair of model and solution

$ns$ pairs of models and solutions

Step 2

- $i = 0$
- $i = i + 1$
- $i < m$
- $i \geq m$

Grid search inversion for best $w_i$ minimizing equation (22)

Yes

No

End

(similar to the left flow-chart)

Figure 4
Figure 5

(a) Resolution matrix

(b) Normalized resolution matrix

Parameter index (i) and (j)
Figure 9